Q - Covariance / Invariance of Maxwell equation in terms of four vectors.

Ans.

The covariance of Maxwell's equation means that the equation have the same form in any two inertial systems.

The Maxwell's field equations in terms of scalar and vector potential can be written as

$$\square^2 \phi = -\rho/\epsilon_0 \longrightarrow (1)$$

& $$\square^2 A = -\mu_0 J \longrightarrow (11)$$

with Lorentz Condition $\nabla \cdot A + \frac{1}{c^2} \cdot \frac{\partial \phi}{\partial t} = 0 \longrightarrow (3)$

The electromagnetic four vector potential A_μ and current fourvector J_μ are represented as

$$A_\mu = \left(A, \frac{i\phi}{c}\right) \longrightarrow$$

$$A_\mu = (A_1, A_2, A_3, A_4) \quad \text{where } A_4 = \frac{i\phi}{c} \longrightarrow (4)$$

and $$J_\mu = (j, ic\rho)$$

$$J_\mu = (j_1, j_2, j_3, j_4) \quad j_4 = ic\rho \longrightarrow (5)$$

The four transformation equation are as follows.

$$A_1' = A_1 \qquad j_1' = J_1$$
$$A_2' = A_2 \qquad J_2' = J_2$$
$$A_3' = \gamma\left(A_3 - \frac{v\phi}{c^2}\right) \qquad J_3' = \gamma(J_3 - v\rho)$$
$$\phi' = \gamma(\phi - vA_3) \qquad \rho' = \gamma(\rho - vJ_3/c^2)$$

In terms of A_μ and J_μ the Maxwell's field equations are expressed in the form

$$\Box^2 A_\mu = -\mu_0 J_\mu \longrightarrow (6)$$

The Lorentz Condition $\sum_{\mu=1}^{4} \frac{\partial A_\mu}{\partial x_\mu} = 0 \longrightarrow (7)$

The equation (6) and (7) are in S frame.

The covariance of Maxwell field equations requires that in any inertial frame S', equation (6) and (7) must have the same form. ie

$$\Box^2 A_\mu' = -\mu_0 J_\mu' \longrightarrow (8)$$

with Lorentz Condition

$$\sum_{\mu=1}^{4} \frac{\partial A_\mu'}{\partial x_\mu'} = 0 \longrightarrow (9)$$

Now $\Box'^2 A_1' = \Box^2 A_1$ as $\Box'^2 = \Box^2$

$$= -\mu_0 J_1'$$

$$\Box'^2 A_1' = -\mu_0 J_1'$$

$$\Box^2 A_2' = \Box^2 A_2$$
$$= -\mu_0 J_2$$

$$\Box'^2 A_3' = \Box^2\left(\gamma\left(A_3 - \frac{v\phi}{c^2}\right)\right)$$
$$= \gamma\left[\Box^2 A_3 - \frac{v}{c^2}\Box^2\phi\right]$$

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$$= \gamma\left[-\mu_0 J_3 - \frac{v}{c^2}\left(-\frac{\rho}{\epsilon_0}\right)\right] \text{ using 2}$$

$$= -\gamma\left[\mu_0 J_3 - \frac{v\rho}{c^2\epsilon_0}\right] \qquad \boxed{c^2 = \frac{1}{\mu_0 c^2}}$$

$$= -\gamma\left[\mu_0 J_3 - \mu_0 v\rho\right]$$

$$= -\mu_0\gamma\left[J_3 - v\rho\right]$$

$$\Box'^2 A_3' = -\mu_0 J_3' \longrightarrow (2)$$

$$\Box'^2 A_4' = \Box^2 \frac{i\phi'}{c}$$

$$= \Box^2 \frac{i}{c}\gamma(\phi - vA_3)$$

$$= \frac{i\gamma}{c}\left[\Box^2\phi - \Box^2 vA_3\right]$$

$$= \frac{i\gamma}{c}\left[-\frac{\rho}{\epsilon_0} - v\Box^2 A_3\right]$$

$$= \frac{i\gamma}{c}\left[-\frac{\rho}{\epsilon_0} - v(-\mu_0 J_3)\right]$$

$$= \frac{-i\gamma}{c}\left[\frac{\rho}{\epsilon_0} - v\mu_0 J_3\right]$$

$$= -\frac{i\gamma}{c}\left[\frac{\rho}{\epsilon_0} - \frac{vJ_3}{c^2\epsilon_0}\right]$$

$$= -\frac{i\gamma}{c\epsilon_0}\left[\rho - \frac{vJ_3}{c^2}\right]$$

$$= \frac{-i\rho'}{c\epsilon_0} = -i\mu_0 c\rho'$$

$$= -\mu_0(ic\rho')$$

$$J_4' = ic\rho$$

$$\Box'^2 A_4' = -\mu_0 J_4'$$

$$A_{\mu}' = \sum_{\nu} a_{\mu\nu} A_{\nu}$$

$$x_{\mu}' = \sum_{\nu} a_{\mu\nu} x_{\nu}$$

Hence $\frac{\partial x_{\mu}'}{\partial x_{\nu}} = a_{\mu\nu}$

$$\therefore \frac{\partial A_{\mu}'}{\partial x_{\mu}'} = \sum a_{\mu\nu}\frac{\partial A_{\nu}}{\partial x_{\mu}}$$

$$= \sum_{\mu\nu}\frac{\partial x_{\mu}'}{\partial x_{\nu}}\frac{\partial A_{\nu}}{\partial x_{\mu}'}$$

$$= \sum_{\nu}\frac{\partial A_{\nu}}{\partial x_{\nu}} = \sum_{\mu}\frac{\partial A_{\mu}}{\partial x_{\mu}}$$

But $\sum_{\mu}\frac{\partial A_{\mu}}{\partial x_{\mu}} = 0$ Hence

$$\sum_{\mu}\frac{\partial A_{\mu}'}{\partial x_{\mu}'} = 0$$

This means that the Maxwell equations are covariant under Lorentz transformation.